

Consistent initial data for CMD perturbations

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Abstract

We investigate the initial condition which are simultaneously consistent with perturbation equations in both the radiation-dominated and the matter-dominated epochs. The exact formula for the spectrum transfer-function is derived.

1 Introduction

The scale invariance of the perturbations growth in the matter dominated epoch is known as the result of both the Newtonian and Einstein gravity theories. It is caused by the absence of the Euler term in the propagation equations [1, 2]. Partial differential equations derived as the first order expansion of Einstein equations, immediately reduce to ordinary ones. According to standard procedures the initial conditions for these equations are being imposed at the beginning of the matter dominated epoch. Initial velocity field is set to zero throughout the space, or alternatively, the decaying mode is neglected [3]. The perturbation then is the product of time-dependent factor and an arbitrary function of the space coordinates. Perturbations of any shape and any scale grow equally slow [1, 4, 5] — too slow to form compact objects in the time interval admitted by observations [6, 7, 8].

Similar constraints for initial conditions are often applied to the n -body simulations [9]. Following [10] numerous treatments additionally adopt the Zeldovich approximation [11] or its generalizations [12]. The perturbations statistics and amplitudes are confronted with the CMB data and inflationary scenarios [13, 14]. Most approaches adopt Newtonian gravity or the Newtonian limit of GR. The n -body simulations are promissory in non-linear regime. The question how to reach nonlinear regime still remains difficult for both simulations and analytical solutions.

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In this paper we investigate initial condition which are simultaneously consistent with the perturbation equations in both, the radiation- and mater-dominated epochs. Following the techniques elaborated in the gravitational waves theory [15, 16, 17, 18, 19, 20], we treat both epochs (with the perturbation corrections included) as the single Riemannian manifold. Appropriate junction conditions guarantee that the first and the second fundamental forms at the decoupling hypersurface Σ_d are continuos. The perturbation modes are unique and regular in the entire spacetime. The momentum spectrum of sound in the radiation-dominated epoch (the gauge-invariant and time-independent Fourier coefficients A_k) uniquely determine the perturbations evolution, both prior and past to decoupling. Due to junction conditions the perturbations in the late epoch ($p = 0$) inherit some properties from the epochs before decoupling. One of the symptoms is the breakdown of the late-time scale-invariance.

2 Brief description of the method

Einstein equations and Darmois–Israel conditions provide solution for the background scale factor

$$a_{(1)}(\eta) = \sqrt{\frac{\mathcal{M}}{3}} \eta, \quad (1)$$

$$a_{(2)}(\eta) = \frac{1}{4} \sqrt{\frac{\mathcal{M}}{3}} \frac{(\eta + \eta_d)^2}{\eta_d}. \quad (2)$$

η_d stands for the time of decoupling, \mathcal{M} is a constant of motion, $\mathcal{M} = a \epsilon^4$ in the radiation filled universe. In the conformal Newtonian gauge with the metric tensor

$$\mathbf{g}_{\mu\nu} = a^2(\eta) \text{diag}(-1 - 2\phi, 1 - 2\phi, 1 - 2\phi, 1 - 2\phi). \quad (3)$$

the propagation equations for the density perturbations in both epochs take forms

$$\frac{1}{\eta} \frac{\partial}{\partial \eta} \left[\frac{1}{\eta^2} \frac{\partial}{\partial \eta} \eta^3 \phi_{(1)}(\eta, x) \right] - \frac{1}{3} \nabla_\mu \nabla^\mu \phi_{(1)}(\eta, x) = 0, \quad (4)$$

for the radiation-dominated epoch, and

$$\frac{\partial}{\partial \eta} \left[\frac{1}{(\eta + \eta_d)^4} \frac{\partial}{\partial \eta} (\eta + \eta_d)^5 \phi_{(2)}(\eta, x) \right] = 0. \quad (5)$$

for the matter-dominated one.

New perturbation variable Ψ defined as the Darboux transform [21] of the potential $\phi_{(1)}(\eta, x)$

$$\Psi(\eta, x) = \frac{1}{\eta} \frac{\partial}{\partial \eta} [\eta^3 \phi_{(1)}(\eta, x)], \quad (6)$$

allows to reduce (4) to d'Alembert equation¹

$$\left[\frac{\partial^2}{\partial \eta^2} - \frac{1}{3} \nabla_\mu \nabla^\mu \right] \Psi(\eta, x) = 0. \quad (7)$$

Ψ is an analogue of the massless scalar field. The general solution to (7) is the linear combination of running waves

$$\Psi(\eta, x) = \sum_k \mathcal{A}_k \mathbf{u}_k(\eta, x) + \text{c.c.}, \quad (8)$$

where the modes

$$\mathbf{u}_k(\eta, x) = \frac{1}{\sqrt{2\pi}} \exp[\mathbf{i}(kx - \omega\eta)] \quad (9)$$

are harmonic in η , and normalized

$$\langle \mathbf{u}_k(\eta, x), \mathbf{u}_{k'}(\eta, x) \rangle = \omega(k) \delta^3(k - k') \quad (10)$$

according to the Klein–Gordon scalar product

$$\langle \Psi_1(\eta, x), \Psi_2(\eta, x) \rangle = \mathbf{i} \int [\Psi_1 \partial_\mu \Psi_2^* - \Psi_2 \partial_\mu \Psi_1^*] d\sigma^\mu. \quad (11)$$

The integral in formula (11) runs over arbitrary space-like hypersurface σ^μ , therefore, the coefficients A_k are gauge-invariant and constant in time. Now, the field ϕ in the radiation-dominated epoch is a combination

$$\phi_{(1)}(\eta, x) = \sum_k \mathcal{A}_k u_{\phi_{(1)}}(\eta, x) + \text{c.c.} \quad (12)$$

of modes

$$u_{\phi_{(1)}}(\eta, x) = \frac{1}{\sqrt{2\pi}} \frac{1}{(\omega\eta)^2} \left[1 + \frac{1}{\mathbf{i}\omega\eta} \right] \exp[\mathbf{i}(kx - \omega\eta)]. \quad (13)$$

In the matter dominated universe equation (5) is solved by an arbitrary combination of the constant or decreasing modes

$$\phi_{(2)}(\eta, x) = \sum_k \left[\alpha_k - \frac{\beta_k}{5(\eta + \eta_d)^5} \right] \exp[\mathbf{i}kx] + \text{c.c.} \quad (14)$$

¹For other gauges or gauge-invariant variables the equivalent procedures are given in [22, 23, 24, 25].

To assure the continuity of the first and the second fundamental forms at the decoupling hypersurface the coefficients \mathcal{A}_k , α_k and β_k must be related by

$$\alpha_k = -\frac{1}{5\sqrt{2\pi}} \left[\frac{1}{(\omega\eta_d)^2} + i \left(\frac{2}{\omega\eta_d} - \frac{1}{(\omega\eta_d)^3} \right) \right] \mathcal{A}_k \exp[-i\omega\eta_d], \quad (15)$$

$$\beta_k = -\frac{64}{\sqrt{2\pi}} \eta_d^5 \left[\frac{3}{(\omega\eta_d)^2} + i \left(\frac{1}{\omega\eta_d} - \frac{3}{(\omega\eta_d)^3} \right) \right] \mathcal{A}_k \exp[-i\omega\eta_d]. \quad (16)$$

Then the generic solution after decoupling is

$$\phi_{(2)} = \sum_k \mathcal{A}_k u_{\phi_{(2)}}(\eta, x) + \text{c.c.} \quad (17)$$

where

$$\begin{aligned} u_{\phi_{(2)}} = & \frac{1}{5\sqrt{2\pi}} \left\{ \frac{1}{(\omega\eta_d)^2} \left[6 \left(\frac{2}{1+\eta/\eta_d} \right)^5 - 1 \right] + i \left[\frac{2}{\omega\eta_d} \left[\left(\frac{2}{1+\eta/\eta_d} \right)^5 - 1 \right] \right. \right. \\ & \left. \left. + \frac{1}{(\omega\eta_d)^3} \left[1 - 6 \left(\frac{2}{1+\eta/\eta_d} \right)^5 \right] \right] \right\} \exp[i(kx - \omega\eta_d)]. \end{aligned} \quad (18)$$

The pair of solutions (13) and (18) combines into a single mode $\{u_{\phi_{(1)}}, u_{\phi_{(2)}}\}$ regular everywhere in the space-time. The density contrast and the four-velocity corrections read

$$\frac{\delta\epsilon}{\epsilon} = 2 \left[-\phi + \frac{1}{3} \left(\frac{a'(\eta)}{a(\eta)} \right)^{-2} \Delta\phi - \left(\frac{a'(\eta)}{a(\eta)} \right)^{-1} \frac{\partial\phi}{\partial\eta} \right] \quad (19)$$

$$\delta U^\mu = \left(\phi, \frac{2}{3} a'(\eta)^{-2} \frac{\partial}{\partial\eta} (a(\eta) \nabla\phi) \right) \quad (20)$$

with ϕ given by (12)–(13) or (17)–(18) respectively. The freedom to choose \mathcal{A}_k is the only freedom left. Initial data on Σ_d determined by (15)–(16) are consistent with both equations, (4) and (5). While evolved forward or backward in time they restore the perturbation field in each of considered epochs.

3 The transfer function

We introduce dimensionless variables κ , χ and τ based on the characteristic time-scale η_d (time of decoupling) and a characteristic length-scale $\eta_d/\sqrt{3}$ (length of the sound horizon)

$$\kappa = \frac{\eta_d}{\sqrt{3}} k, \quad \chi = \frac{x}{\eta_d/\sqrt{3}}, \quad \tau = \frac{\eta}{\eta_d}. \quad (21)$$

κ and τ relate to observables

$$\tau = \frac{2\sqrt{1+z_d}}{\sqrt{1+z}} - 1 \quad (22)$$

$$\kappa = \left(\frac{4\pi^3}{3\sqrt{3}GH_0\mathcal{M}} \right)^{1/3} (1+z_d)^{-1/2} \quad (23)$$

where H_0 stands for the Hubble constant, z_d is the redshift of decoupling, \mathcal{M} is the mass of sphere of the wavelength diameter.

The density perturbations expressed in dimensionless variables take the form

$$\delta_{(1)} = \sum_k \mathcal{A}_k u_{\delta_{(1)}}(\tau, \chi) + \text{c.c.} \quad (24)$$

$$\delta_{(2)} = \sum_k \mathcal{A}_k u_{\delta_{(2)}}(\tau, \chi) + \text{c.c.} \quad (25)$$

where

$$u_{\delta_{(1)}} = \frac{-2\mathbf{i}\kappa\tau}{(\kappa\tau)^4} (2 + \mathbf{i}\kappa\tau(2 + \mathbf{i}\kappa\tau(2 + \mathbf{i}\kappa\tau))) \exp(\mathbf{i}\kappa(\chi - \tau)) \quad (26)$$

and

$$\begin{aligned} u_{\delta_{(2)}} = & -\frac{\mathbf{i}\kappa}{\kappa^4} \exp(\mathbf{i}\kappa(\chi - 1)) \left[2 \left[\frac{2}{1+\tau} \right]^5 (3 + \mathbf{i}\kappa(3 + \mathbf{i}\kappa)) - \frac{2}{5} \left[\kappa^2 \left[\frac{1+\tau}{2} \right]^2 + 1 \right] \right. \\ & \times \left. \left[1 + \mathbf{i}\kappa(1 + 2\mathbf{i}\kappa) - 2 \left[\frac{2}{1+\tau} \right]^5 (3 + \mathbf{i}\kappa(3 + \mathbf{i}\kappa)) \right] \right]. \end{aligned} \quad (27)$$

The linear dispersion relation $\omega = k/\sqrt{3}$ is fulfilled. Perturbations of both the energy density and the expansion rate are continuous at the transition.

The transfer function [26] normalized to the decoupling epoch $\tau = 1$ reads

$$T(\kappa, \tau) = \frac{\sqrt{u_{\delta_{(2)}}(\tau, \chi) u_{\delta_{(2)}}^*(\tau, \chi)}}{\sqrt{u_{\delta_{(2)}}(1, \chi) u_{\delta_{(2)}}^*(1, \chi)}}. \quad (28)$$

On the strength of (27) one has

$$\begin{aligned} T^2(\kappa, \tau) = & \frac{1}{800(2 - 2\kappa^2 + \kappa^4)} [(9 + 3\kappa^2 + \kappa^4)f^2 - 2(3 - 4\kappa^2 + \kappa^4)gf + (1 - 3\kappa^2 + 4\kappa^4)g^2] \\ f = & 2(\kappa^2(1 + \tau)^2 - 6) \left(\frac{1 + \tau}{2} \right)^{-5} \\ g = & 4 + \kappa^2(1 + \tau)^2 \end{aligned} \quad (29)$$

and therefore,

$$a_k(\tau)a_k^*(\tau) = T^2(\kappa, \tau)a_k(1)a_k^*(1) \quad (30)$$

with the usual definition of a_k given by

$$\begin{aligned} \delta_{(2)}(\tau, \chi) &= \sum_k \mathcal{A}_k u_{\delta_{(2)}}(\tau, \chi) + \text{c.c.} \\ &= \sum_k a_k(\tau) \exp(\mathbf{i}\kappa\chi) + \text{c.c.} \end{aligned} \quad (31)$$

The transfer function (29) differentiate perturbations of various length scales. The contour plot of $T(\mathcal{M}, z)$ as the function of mass and redshift (in the logarithmic scale) is shown on Figure 1. The mass \mathcal{M} is expressed in the solar mass units, the decoupling redshift $z_d = 1200$ is assumed. Conditions (15) and (16) do not restrain the perturbations spectrum. The invariant spectrum $A_k A_k^*$ (or equivalently the coefficients $a_k(t_d)$ on Σ_d) remains a free parameter of the model.

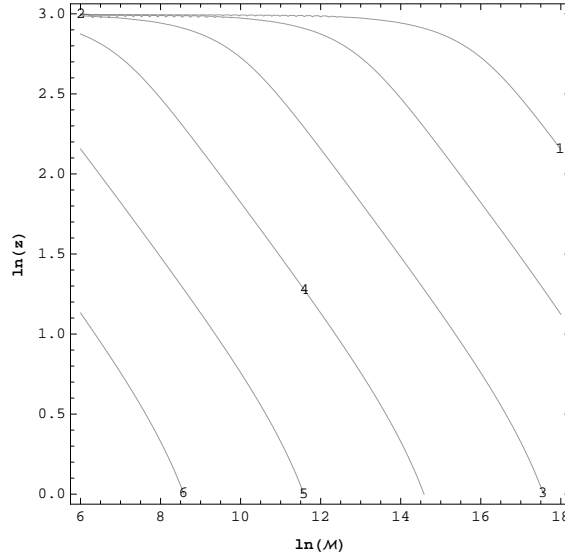


Figure 1: Logarithmic contour map of the transfer function $\ln T(\ln \mathcal{M}, \ln z)$. z stands for redshift, \mathcal{M} is the perturbation mass expressed in the Solar mass units.

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